Degree sequences related to degree set : extended abstract (Mathematical Foundations and Applications of Computer Science and Algorithms)

Osawa, Shingo; Shibata, Yukio

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Degree sequences related to degree set (extended abstract)

Shingo Osawa* Yukio Shibata*

Department of Computer Science, Gunma University,
1-5-1 Tenjin-cho, Kiryu, Gunma, 376-8515 Japan

1 Introduction

A sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers is called *graphical* if there exists a simple graph G with vertices $v_1, v_2, \ldots, v_n$ such that v_k has degree d_k for each k. Certainly the conditions $d_k \leq n-1$ for all k and $\sum_{k=1}^{n} d_k$ being even are necessary for a sequence to be graphical, but these conditions are not sufficient. A neccessary and sufficient conditon for a sequence to be graphical was found by Havel [4] and later rediscoverd by Hakimi [3]. Another characterization that determines which sequences are graphical is due to Erdös and Gallai [1]. Recently, Tripathi and Vijay [8] improved the result of Erdös and Gallai.

The degree set of a graph G is the set $\mathcal{D}$ consisting of the distinct degrees of vertices in G. The question which sets of positive integers are the degree sets of graphs has been investigated. Kapoor, Polimeni and Wall [7] completely answered that question, and Tripathi and Vijay [10] have given a short proof for the theorem of them. Recently, Tripathi and Vijay [9] have given the new result on the graph with the least order and the least size among graphs with the given degree set.

We propose a basic ploblem. If a degree sequence is given, we immediately obtain a unique degree set. Conversely, if a degree set is given, we wonder whether there is a procedure to obtain graphical degree sequences with the given degree set. In general, there are infinitely many graphs with the given degree set. Even if the order is restricted to the least one, we might find several graphs with the given degree set.

Here, we are in the state to propose the following problem.

Problem For a given degree set, how many degree sequences with the least order are there?

2 Degree 2-set

Let p and q be the number of vertices and the number of edges of the graph G, respectively. Let $\mathcal{D} = \{a, b\}$ be a degree set, where $a, b > 0$, and $a > b$. We shall employ the notation $(c)_m$ to denote m occurrence of the integer c in the degree sequence. We may denote a degree sequence by $s = (a)_x(b)_y$, where $a > b$, and $x, y > 0$ with $x + y = p$.

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Then we obtain the following equation;

$$\begin{cases} ax + by = 2q, \\ x + y = p, \\ x, y > 0. \end{cases} \tag{1}$$

Since x and y are positive integer solutions, we obtain next necessay conditions

$$\begin{cases} (a-b)|(2q-bp), \\ (a-b)|(ap-2q), \\ 2q-bp > 0, \\ ap-2q > 0. \end{cases} \tag{2}$$

Under the conditions (2), we have the solution for the equation (1) as follows,

$$\begin{cases} x = \frac{2q-bp}{a-b}, \\ y = \frac{ap-2q}{a-b}. \end{cases} \tag{3}$$

Because of a condition for graphs, $p \geq a+1$. Then we consider a graph with order $p = a+1$. We set $p = a+1$ for the conditions (2) and obtain the following conditions;

$$\begin{cases} (a-b)|2q-b(a+1), \\ (a-b)|a(a+1)-2q, \\ 2q-b(a+1) > 0, \\ a(a+1)-2q > 0. \end{cases} \tag{4}$$

From the third and the fourth inequalities of the equation in (4), we secure the bound for q, that is $\frac{b(a+1)}{2} < q < \frac{a(a+1)}{2}$.

We have the solution under the conditions (4),

$$\begin{cases} x = \frac{2q-b(a+1)}{a-b}, \\ y = \frac{a(a+1)-2q}{a-b}. \end{cases} \tag{5}$$

For convenience, we may write $a = k+l$ and $b = k$, and we have

$$\begin{cases} x = \frac{2q-k(k+l+1)}{l}, \\ y = \frac{(k+l)(k+l+1)-2q}{l}, \\ \frac{k(k+l+1)}{2} < q < \frac{(k+l)(k+l+1)}{2}. \end{cases} \tag{6}$$

Lemma 2.1 *The number of vertices with the maximum degree $k+l$ is less than or equal to the minimum degree k, that is $x \leq k$.*

Lemma 2.2 *The number of edges q is less than or equal to $k(k+2l+1)/2$, that is $q \leq k(k+2l+1)/2$.*

By Lemma 2.2, we rewrite the equation (6) as follows;

$$\begin{cases} x = \frac{2q-k(k+l+1)}{l}, \\ y = \frac{(k+l)(k+l+1)-2q}{l}, \\ \frac{k(k+l+1)}{2} < q \leq \frac{k(k+2l+1)}{2}. \end{cases} \tag{7}$$

Theorem 2.1 *Let $\mathfrak{D} = \{k+l, k\}$ be a degree set, where $k > 0$ and $l > 0$. Then, we have the following properties:*

1. *k and l are even.*

   There exist k degree sequences $s = (k+l)_x(l)_y$ for each $(x, y) = (1, k+l)$, $(2, k+l-1), \ldots, (k, l+1)$. Moreover if $(x, y) = (1, k+l)$, a number of edges $q = \frac{(k+1)(k+l)}{2}$ is minimum, and if $(x, y) = (k, l+1)$, $q = \frac{k(k+2l+1)}{2}$ is the maximum.

2. *k is even and l is odd.*

   There exist $\frac{k}{2}$ degree sequences $s = (k+l)_x(l)_y$ for each $(x, y) = (2, k+l-1)$, $(4, k+l-3), \ldots, (k, l+1)$. Moreover if $(x, y) = (2, k+l-1)$, a number of edges $q = \frac{k(k+l+1)}{2} + l$ is minimum, and if $(x, y) = (k, l+1)$, $q = \frac{k(k+2l+1)}{2}$ is maximum.

3. *k is odd and l is even.*

   There exist k degree sequences $s = (k+l)_x(l)_y$ for each $(x, y) = (1, k+l)$, $(2, k+l-1), \ldots, (k, l+1)$. Moreover if $(x, y) = (1, k+l)$, a number of edges $q = \frac{(k+1)(k+l)}{2}$ is minimum, and if $(x, y) = (k, l+1)$, $q = \frac{k(k+2l+1)}{2}$ is maximum.

4. *k and l are odd.*

   There exist $\lceil \frac{k}{2} \rceil$ degree sequences $s = (k+l)_x(l)_y$ for each $(x, y) = (1, k+l)$, $(3, k+l-2), \ldots, (k, l+1)$. Moreover if $(x, y) = (1, k+l)$, the number of edges $q = \frac{k(k+l+1)}{2} + \frac{l}{2}$ is minimum, and if $(x, y) = (k, l+1)$, $q = \frac{k(k+l+1)}{2} + \frac{k}{2}l$ is maximum.

If $\mathfrak{D} = \{k+l, k\}$, then KPW algorithm produces a graph $K_k + \overline{K_{l+1}}$, while TV algorithm generates a graph $\overline{K_{l+1} \cup \overline{K_k}}$. These graphs are isomorphic and have $k(k+2l+1)/2$ edges, and the number of edges is maximum.

Cororally 2.1 *Let $\mathfrak{D} = \{k, 1\}$ be a degree set, where $k \geq 2$. Then the graph with the degree set $\mathfrak{D}$ is unique.*

Cororally 2.2 *Let $\mathfrak{D} = \{2k+1, 2\}$ be a degree set, where $k \geq 1$. Then the graph with the degree set $\mathfrak{D}$ is unique.*

Therefore, the graph generated by KPW algorithm and TV algorithm is the above one.

Cororally 2.3 *Let $\mathfrak{D} = \{2k, 2\}$ be a degree set, where $k \geq 2$. There exist two graphs with the degree set $\mathfrak{D}$.*

The graph generated by KPW algorithm and TV algorithm has the degree sequence $s = (2k)_2(2)_{2k-1}$, and the number of edges $q = 4k - 1$ is maximum.

2.1 Example

Let $\mathfrak{D} = \{9, 4\}$ be a degree set. We have $k = 4$ and $l = 5$. By Theorem 2.1, we obtain two degree sequences, $(9)_2(4)_8$ and $(9)_4(4)_6$. The sequence $(9)_4(4)_6$ has unique graph, while another has three nonisomorphic graphs $K_2 + C_8$, $K_2 + 2C_4$ and $K_2 + (C_3 \cup C_5)$.

3 Degree n-set

Let p and q be the number of vertices and the number of edges of the graph G, respectively. Let $\mathcal{D} = \{d_1, d_2, \ldots, d_n\}$ be a degree set, where $d_1 > d_2 > \cdots > d_n > 0$. We shall employ the notation $(c)_m$ to denote m occurrence of the integer c in the degree sequence. We may denote a degree sequence by $s = (d_1)_{m_1}(d_2)_{m_2}, \ldots, (d_n)_{m_n}$, where $d_1 > d_2 > \cdots > d_n > 0$ and $m_i > 0$, $1 \leq i \leq n$, with $m_1 + m_2 + \cdots + m_n = p$.

Then we obtain the following equation;

$$\begin{cases} d_1 m_1 + d_2 m_2 + \cdots + d_n m_n = 2q, \\ m_1 + m_2 + \cdots + m_n = p = d_1 + 1, \\ m_i > 0, i = 1, 2, \ldots, n. \end{cases} \tag{8}$$

If we have positive integers $m_i, i = 1, 2, \ldots, n$ and q which satisfy the Diophantine equation (8), candidates for the number of edges and degree sequences with a given degree set $\mathcal{D}$ could be found.

In order to find $m_i, i = 1, 2, \ldots, n$, we introduce indefinite equations and we replace $m_i, i = 1, 2, \ldots, n$ in the equation (8) by $x_i, i = 1, 2, \ldots, n$, respectively.

$$\begin{cases} d_1 x_1 + d_2 x_2 + \cdots + d_n x_n = 2q, \\ x_1 + x_2 + \cdots + x_n = p = d_1 + 1. \end{cases} \tag{9}$$

Substituting from

$$x_n = d_1 + 1 - x_1 - x_2 - \cdots - x_{n-1}$$

to the first equation in (13), we obtain the following equation:

$$\begin{aligned} & (d_1 - d_n)x_1 + (d_2 - d_n)x_2 + \cdots + (d_{n-1} - d_n)x_{n-1} \\ = \; & 2q - d_n(d_1 + 1). \end{aligned} \tag{10}$$

Let $g = \gcd((d_1 - d_n), (d_2 - d_n), \cdots, (d_{n-1} - d_n))$. By the chinese remainder theorem, there exist integers $x_i, i = 1, 2, \ldots, n-1$, which satisfy equation (10) if and only if $g | 2q - d_n(d_1 + 1)$.

Therefore, candidates for the number of edges with the degree set $\mathcal{D}$ satisfy the condition $2q - d_n(d_1 + 1) = kg$ for some integer k and are obtained as follows;

$$q = \frac{kg + d_n(d_1 + 1)}{2}. \tag{11}$$

Let $g * a_i = d_i - d_n, i = 1, 2, \ldots, n-1$, and we use $kg = 2q - d_n(d_1 + 1)$, then the equation (10) is expressed by

$$g * a_1 x_1 + g * a_2 x_2 + \cdots + g * a_{n-1} x_{n-1} = kg.$$

Hence, if we find integer solutions $x_i, i = 1, 2, \ldots, n-1$ of

$$a_1 x_1 + a_2 x_2 + \cdots + a_{n-1} x_{n-1} = k, \tag{12}$$

then we secure candidates for a multiple number of each degree. Since $m_i > 0$, $i = 1, 2, \ldots, n$, only positive solutions are the candidates for degree sequence. Equivalently,

$$\begin{cases} a_1 x_1 + a_2 x_2 + \cdots + a_{n-1} x_{n-1} = k, \\ x_i > 0, i = 1, 2, \ldots, n-1. \end{cases} \tag{13}$$

This is an integer knapsack problem which is known NP-complete. However, this problem is solvable in pseudo-polynomial time by dynamic programming ([2],[6]).

4 Bounds for k

First, we evaluate a lower bound for k.

Lemma 4.1

$$\frac{1}{g}\sum_{i=1}^{n-1}(d_i - d_n) \leq k.$$

We show the next lemma.

Lemma 4.2 *The number of vertices with the maximum degree d_1 is less than or equal to the minimum degree d_n, that is $m_1 \leq d_n$.*

Next, we wonder how large the parameter k is, and try to evaluate it.

Lemma 4.3

$$k \leq \frac{1}{g}\left\{\sum_{i=1}^{n} d_i + (d_2 - 1)d_1 - \{d_n + (n-2)\}d_2 - d_n\right\}.$$

From Lemma 4.1 and Lemma 4.3, we obtain the bounds for k;

Theorem 4.1

$$\frac{1}{g}\sum_{i=1}^{n-1}(d_i - d_n) \leq k \leq \frac{1}{g}\left\{\sum_{i=1}^{n} d_i + (d_2 - 1)d_1 - \{d_n + (n-2)\}d_2 - d_n\right\}.$$

4.1 Example

We consider degree sequences for a degree set $\mathcal{D} = \{8, 6, 4, 2\}$.

We substitute the degree set $\mathcal{D} = \{8, 6, 4, 2\}$ and $\gcd(8-2, 6-2, 4-2) = 2$ to the equation (13), then we have the equation

$$6x_1 + 4x_2 + 2x_3 = 2k,$$

where k is a parameter. This equation is reduced to

$$3x_1 + 2x_2 + x_3 = k. \tag{14}$$

We solve the equation (14).

We substitute a degree set $\mathcal{D} = \{8, 6, 4, 2\}$ and $\gcd(8-2, 6-2, 4-2) = 2$ to Theorem 4.1, then

$$k \leq \frac{1}{2}\{8 + 6 + 4 + 2 + (6-1) * 8 - (2 + 4 - 2) * 6 - 2\} = 17,$$

and

$$\frac{1}{2}\{(8-2) + (6-2) + (4-2)\} = 6 \leq k.$$

Therefore, it is enough to solve the equation (14) for $6 \leq k \leq 17$.

If $k = 6$, then we obtain a unique positive solution $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, and $x_4 = 6$. Next we check whether the obtained candidate $8, 6, 4, 2, 2, 2, 2, 2, 2$ is graphical, and it is true.

If $k = 17$, we obtain a positive solution $x_1 = 2, x_2 = 5, x_3 = 1, x_4 = 1$, then we have the candidate $8, 8, 6, 6, 6, 6, 6, 4, 2$, and it is graphical.

Our bounds are sharp.

5 Enumeration of candidates for degree sequence

To solve all solutions for the equation (13), we introduce a candidate tree. Each vertex in the candidate tree is an n-tuple, and each n-tuple corresponds to a solution for the equation (13). If k is the lower bound for Theorem 4.1, then the equation (13) has a solution $(1,1,\ldots,1,d_1-n+2)$. We set a root to $(1,1,\ldots,1,d_1-n+2)$, and the root has $n-1$ children whose n-tuple are $(2,1,\ldots,1,d_1-n+1)$, $(1,2,1,\ldots,1,d_1-n+1),\ldots,(1,\ldots,1,2,d_1-n+1)$. The vertex $(1,1,\ldots,1,\alpha,\beta_1,\ldots,\beta_m,d_1-n+2-k)$ has $n-m-1$ children whose n-tuple are $(2,1,\ldots,1,\alpha,\beta_1,\ldots,\beta_m,d_1-n+2-k-1)$, $(1,2,\ldots,1,\alpha,\beta_1,\ldots,\beta_m,d_1-n+2-k-1),\ldots(1,1,\ldots,1,\alpha+1,\beta_1,\ldots,\beta_m,d_1-n+2-k-1)$, where $\alpha>1$ is an integer, $\beta_i>0$ is an integer, $1\le i\le m<n$, and $k<d_1-n+2$ is some positive integer. Moreover, the vertices $(\delta_1,\ldots,\delta_{n-1},1)$ have no child, where $\delta_i\le d_n$ is an integer and $\delta_i>0$ is an integer, $2\le i<n$. And the vertices $(d_n,\gamma_1,\cdots,\gamma_{n-1})$ have no child, where $\gamma_i>0$ is an integer $1\le i<n$, because by Lemma 4.2, if $x_1>d_n$, there is no solution for the equation (13). In other words, the parent of vertex $(1,1,\ldots,1,\alpha,\beta_1,\ldots,\beta_m,d_1-n+2-k)$ is $(1,1,\ldots,1,\alpha-1,\beta_1,\ldots,\beta_m,d_1-n+2-k+1)$, where $\alpha>1$ is an integer, $\beta_i>0$ is an integer, $1\le i\le m<n$, and $k<d_1-n+2$ is some positive integer. So, we may uniquely determine the parent. We obtain the candidate tree, whose height is d_1-n+1. The solution $(d_n,d_1-(n-2),1,\ldots,1)$ for the equation (14), which satisfy the upper bound of Theorem 4.3, appears at depth d_1-n+1 in the candidate tree.

Let $NV_i(k)$ denotes the number of vertices whose coordinate i in the n-tuple precisely increases by one at depth k in the tree, where $1\le i\le n-1$ and $0\le k\le d_1-n+1$.

Then we obtain next reccurence,

$$\begin{aligned}
NV_0(0) &= 1,\\
NV_1(1) &= NV_2(1)=\cdots=NV_{n-1}(1)=1,\\
NV_1(k) &= \sum_{j=1}^{n-1} NV_j(k-1)-NV_1(k-d_n+1),\\
NV_i(k) &= \sum_{j=i}^{n-1} NV_j(k-1), \text{where } i,k>1.
\end{aligned} \tag{15}$$

Notice that $\sum_{i=1}^{n-1} NV_i(k)$ denotes the number of vertices at depth k in the tree. Therefore, the total number of vertices in the tree is $\sum_{k=0}^{d_1-n+1}\sum_{i=1}^{n-1} NV_i(k)$.

Rising factorial powers are defined by the rule

$$x^{\overline{n}} = \overbrace{x(x+1)\cdots(x+n-1)}^{n \text{ factors}},$$

where x and n are nonnegative integers.

$NV_{n-1}(k)=1$, $NV_{n-2}(k)=k$, $NV_{n-3}(k)=k(k+1)/2$, $NV_{n-4}(k)=k(k+1)(k+2)/6$, and so on. We have $NV_{n-j}(k)=\frac{k^{\overline{j-1}}}{(j-1)!}$.

Lemma 5.1 *Let x be a positive integer and m a nonnegative integer. Then,*

$$\sum_{i=0}^{m}\frac{(x-1)^{\overline{i}}}{i!}=\frac{x^{\overline{m}}}{m!}.$$

Lemma 5.2 *Let m and n be nonnegative integers. Then,*

$$\sum_{k=0}^{n} k^{\overline{m}} = \frac{n^{\overline{m+1}}}{m+1}.$$

Theorem 5.1 *The candidate tree has*

$$\frac{(d_1 - n + 2)^{\overline{n-1}}}{(n-1)!} - \frac{(d_1 - n + 1 - d_n + 1)^{\overline{n-1}}}{(n-1)!} \tag{17}$$

vertices. That is, the number of candidates for degree sequence is given by the formula in equation (17).

5.1 Example

We consider degree sequences for a degree set $\mathcal{D} = \{8, 6, 4, 2\}$. Then $n = 4$, $d_1 - n + 1 = 5$. We construct the candidate tree for $\mathcal{D} = \{8, 6, 4, 2\}$, and show it in Figure 1.

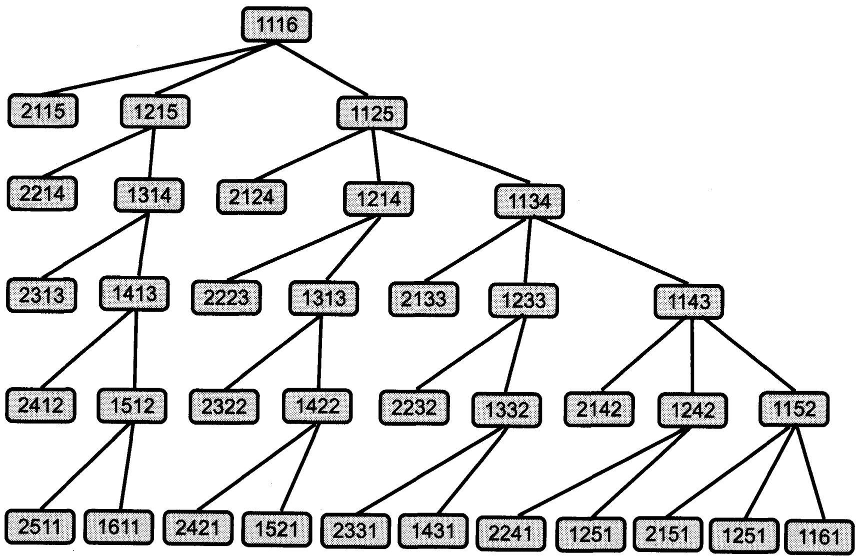

Figure 1: Candidate tree for $\mathcal{D} = \{8, 6, 4, 2\}$.

The total number of vertices in the candidate tree is

$$\begin{aligned} & \frac{(d_1 - n + 2)^{\overline{n-1}}}{(n-1)!} - \frac{(d_1 - n + 1 - d_n + 1)^{\overline{n-1}}}{(n-1)!} \\ = & \frac{6^{\overline{3}}}{3!} - \frac{4^{\overline{3}}}{3!} = \frac{6 \cdot 7 \cdot 8}{3!} - \frac{4 \cdot 5 \cdot 6}{3!} = 56 - 20 = 36 \end{aligned}$$

We exhibit all solutions of the equation (14) in Tabel 1, and check whether the candidates are graphical.

A symbol q in Table 1 denotes the number of edges and is given in the equation (11).

Table 1: Candidates for degree sequences with $\mathcal{D} = \{8, 6, 4, 2\}$

k	x_1	x_2	x_3	x_4	q	graphical?	k	x_1	x_2	x_3	x_4	q	graphical?
6	1	1	1	6	15		12	1	4	1	3	21	No
7	1	1	2	5	16		12	2	1	4	2	21	
8	1	1	3	4	17		12	2	2	2	3	21	No
8	1	2	1	5	17	No	13	1	3	4	1	22	
9	1	1	4	3	18		13	1	4	2	2	22	
9	1	2	2	4	18		13	2	1	5	1	22	
9	2	1	1	5	18	No	13	2	2	3	2	22	
10	1	1	5	2	19		13	2	3	1	3	22	
10	1	2	3	3	19		14	1	4	3	1	23	
10	1	3	1	4	19		14	1	5	1	2	23	
10	2	1	2	4	19	No	14	2	2	4	1	23	
11	1	1	6	1	20		14	2	3	2	2	23	No
11	1	2	4	2	20		15	1	5	2	1	24	
11	1	3	2	3	20		15	2	3	3	1	24	
11	2	1	3	3	20	No	15	2	4	1	2	24	No
11	2	2	1	4	20		16	1	6	1	1	25	
12	1	2	5	1	21		16	2	4	2	1	25	
12	1	3	3	2	21		17	2	5	1	1	26	

6 Conclusion

We propose a basic problem of degree sets. We determine the number of candidates for degree sequences with the least order for a given degree set.

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